Modelling of General Constitutive Relationships in SCN TLM

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ABSTRACT

The modelling of general constitutive relationships in SCN (Symmetrical Condensed Node) TLM is presented. The technique consists of decoupling the scattering matrix from the medium by using equivalent node sources with state-variable formulation of the constitutive relationships. The procedure requires few modifications to TLM. Numerical examples are presented.

INTRODUCTION

The basic TLM formulation requires that the constitutive parameters of the medium ε, μ and σ be specified as constants [1]. Therefore, an appropriate wide-band modelling of general (dispersive, nonlinear, etc...) media cannot be performed without major modification of the algorithm.

The TLM algorithm can be modified to represent general constitutive relationships. This was done in 2D-TLM for frequency dependent [2] and nonlinear [3] dielectric properties by separating the scattering at the node from the properties of the dielectric. The extraction was achieved by representing the dielectric as a circuit network connected to the node by a transmission line of normalized characteristic admittance $Y_r/Y_0=4$.

The procedure presented in this paper is a generalization of this approach for three dimensions with efficient representation of the constitutive relationships. This is done by applying the technique described in [5] to previously decoupled scattering matrices. The medium constitutive relationships are solved using equivalent node sources and state-variables.

THEORY

This section describes the formulation of general materials for SCN TLM modelling. It is organized in two major topics: obtention of the SCN and formulation of state-variable equations. In the first part, decoupled series and shunt nodes are used to obtain the SCN formulation. The second part describes the obtention of the state-equations of the medium.

I - SCN FORMULATION

The SCN formulation is obtained by combining 2D shunt and series nodes [5]. These nodes are decoupled from the medium using the procedure described in [3]. The resulting medium constitutive equations are:

$$
\begin{aligned}
i_{polx} &= 2Y_0\Delta t\frac{dp_x}{dt} & v_{magx} &= 2Z_0\Delta t\frac{dm_x}{dt}\\
i_{poly} &= 2Y_0\Delta t\frac{dp_y}{dt} & v_{magy} &= 2Z_0\Delta t\frac{dm_y}{dt}\\
i_{polz} &= 2Y_0\Delta t\frac{dp_z}{dt} & v_{magz} &= 2Z_0\Delta t\frac{dm_z}{dt}\\
p_x(t) &= P_x(t)\frac{\Delta l}{2\varepsilon_0} & m_x(t) &= M_x(t)\frac{\Delta l}{2}\\
p_y(t) &= P_y(t)\frac{\Delta l}{2\varepsilon_0} & m_y(t) &= M_y(t)\frac{\Delta l}{2}\\
p_z(t) &= P_z(t)\frac{\Delta l}{2\varepsilon_0} & m_z(t) &= M_z(t)\frac{\Delta l}{2}
\end{aligned}
\qquad (1)
$$

where:

$$
\begin{aligned}
i_{polx} &= Y_r(v^r_{ax}-v^i_{ax}) & v_x &= v^r_{ax}+v^i_{ax}\\
i_{poly} &= Y_r(v^r_{ay}-v^i_{ay}) & v_y &= v^r_{ay}+v^i_{ay}\\
i_{polz} &= Y_r(v^r_{az}-v^i_{az}) & v_z &= v^r_{az}+v^i_{az}\\
v_{magx} &= v^r_{nx}+v^i_{nx} & i_x &= (v^r_{nx}-v^i_{nx})/Z_r\\
v_{magy} &= v^r_{ny}+v^i_{ny} & i_y &= (v^r_{ny}-v^i_{ny})/Z_r\\
v_{magz} &= v^r_{nz}+v^i_{nz} & i_z &= (v^r_{nz}-v^i_{nz})/Z_r\\
p_x &= p_x(v_x,v_y,v_z,i_x,i_y,i_z) & m_x &= m_x(v_x,v_y,v_z,i_x,i_y,i_z)\\
p_y &= p_x(v_x,v_y,v_z,i_x,i_y,i_z) & m_y &= m_y(v_x,v_y,v_z,i_x,i_y,i_z)\\
p_z &= p_x(v_x,v_y,v_z,i_x,i_y,i_z) & m_z &= m_z(v_x,v_y,v_z,i_x,i_y,i_z)
\end{aligned}
\qquad (2)
$$

1995 IEEE MTT-S Digest

and the reflected voltages on branches 1-12 are calculated using [5], with the numbering scheme given in [6]:

$$
\begin{aligned}
v_1^r &= v_x(t) - Z_r i_z(t) - v_{12}^i \\
v_2^r &= v_x(t) + Z_r i_y(t) - v_9^i \\
v_3^r &= v_y(t) + Z_r i_z(t) - v_{11}^i \\
v_4^r &= v_y(t) - Z_r i_x(t) - v_8^i \\
v_5^r &= v_z(t) + Z_r i_x(t) - v_7^i(t) \\
v_6^r &= v_z(t) - Z_r i_y(t) - v_{10}^i \\
v_7^r &= v_z(t) - Z_r i_x(t) - v_5^i \\
v_8^r &= v_y(t) + Z_r i_x(t) - v_4^i \\
v_9^r &= v_x(t) - Z_r i_y(t) - v_2^i \\
v_{10}^r &= v_z(t) + Z_r i_y(t) - v_6^i \\
v_{11}^r &= v_y(t) - Z_r i_z(t) - v_3^i \\
v_{12}^r &= v_x(t) + Z_r i_z(t) - v_1^i
\end{aligned} \tag{3}
$$

The equivalent current sources are obtained from the polarization current i_{pol} and the total voltage across the node while the equivalent voltage sources are created with the magnetization voltage v_{mag} and total current at the node. The resulting sources are substituted into (1):

$$
\begin{aligned}
i_{sx}(t) &= 2Y_r v_{ax}^r(t) = 8v_{ax}^r(t) = 2\Delta t \frac{d}{dt} p_x(t) + 4v_x(t) \\
i_{sy}(t) &= 2Y_r v_{ay}^r(t) = 8v_{ay}^r(t) = 2\Delta t \frac{d}{dt} p_y(t) + 4v_y(t) \\
i_{sz}(t) &= 2Y_r v_{az}^r(t) = 8v_{az}^r(t) = 2\Delta t \frac{d}{dt} p_z(t) + 4v_z(t) \\
v_{sx}(t) &= 2Z_r v_{nx}^r(t) = \frac{2}{Z_0} v_{nx}^r(t) = 2\Delta\, t \frac{d}{dt} m_x(t) + 4i_x(t) \\
v_{sy}(t) &= 2Z_r v_{ny}^r(t) = \frac{2}{Z_0} v_{ny}^r(t) = 2\Delta\, t \frac{d}{dt} m_y(t) + 4i_y(t) \\
v_{sz}(t) &= 2Z_r v_{nz}^r(t) = \frac{2}{Z_0} v_{nz}^r(t) = 2\Delta\, t \frac{d}{dt} m_z(t) + 4i_z(t)
\end{aligned} \tag{4}
$$

where:

$$
\begin{aligned}
v_x(t) &= E_x(t)\Delta l \qquad & i_x(t) &= H_x(t)\Delta l \\
v_y(t) &= E_y(t)\Delta l \qquad & i_y(t) &= H_y(t)\Delta l \\
v_z(t) &= E_z(t)\Delta l \qquad & i_z(t) &= H_z(t)\Delta l
\end{aligned} \tag{5}
$$

The sources are calculated at each timestep by:

$$
\begin{aligned}
i_{sx}(t) &= 2(v_1^i(t) + v_2^i(t) + v_9^i(t) + v_{12}^i(t)) \\
i_{sy}(t) &= 2(v_3^i(t) + v_4^i(t) + v_8^i(t) + v_{11}^i(t)) \\
i_{sz}(t) &= 2(v_5^i(t) + v_6^i(t) + v_7^i(t) + v_{10}^i(t)) \\
v_{sx}(t) &= \frac{2}{Z_0}(v_4^i(t) - v_5^i(t) + v_7^i(t) - v_8^i(t)) \\
v_{sy}(t) &= \frac{2}{Z_0}(v_6^i(t) - v_2^i(t) + v_9^i(t) - v_{10}^i(t)) \\
v_{sz}(t) &= \frac{2}{Z_0}(v_1^i(t) - v_3^i(t) - v_{12}^i(t) + v_{11}^i(t))
\end{aligned} \tag{6}
$$

The solution of (5), (4) and (3) will result in the reflected voltages in lines 1-12. The propagation between nodes is not affected.

This procedure is applicable to all kinds of constitutive relationships. The adaptation to usual TLM programs, [6], is done by setting $Y_x=Y_y=Y_z=4$ and $Z_x=Z_y=Z_z=4$ to calculate the equivalent sources (5) and using (4) and (2) to obtain the incident voltages from the stubs. The reflected voltages in branches 1-12 are obtained with (3).

II - STATE-VARIABLE APPROACH

The calculation of the voltages v_x, v_y and v_z and the currents i_x, i_y and i_z can be done once the constitutive relationships **P=P(E,H)** and **M=M(E,H)** are defined. In the state-variable approach all the voltages and currents can be calculated for all kinds of media.

A example is the anisotropic material with non-diagonal tensor:

$$
\begin{aligned}
\boldsymbol{P} &= \varepsilon_0 [F] \boldsymbol{E} = \varepsilon_0 ([\varepsilon_r] - [U]) \boldsymbol{E} \\
\boldsymbol{M} &= [G] \boldsymbol{H} = ([\mu_r] - [U]) \boldsymbol{H}
\end{aligned} \tag{7}
$$

where:

$$
[\varepsilon_r] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}
\qquad
[\mu_r] = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \tag{8}
$$

$$
[U] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

The constitutive state-equation is:

$$2\Delta t\frac{d}{dt}\begin{bmatrix} v_x \\ v_y \\ v_z \\ i_x \\ i_y \\ i_z \end{bmatrix} = -4\begin{bmatrix} [F]^{-1} & [0] \\ [0] & [G]^{-1} \end{bmatrix}\left(\begin{bmatrix} v_x \\ v_y \\ v_z \\ i_x \\ i_y \\ i_z \end{bmatrix} + \frac{2}{Z_0}\begin{bmatrix} v^r_{ax} \\ v^r_{ay} \\ v^r_{az} \\ v^r_{nx} \\ v^r_{ny} \\ v^r_{nz} \end{bmatrix}\right) \qquad (9)$$

The state-variable approach also allows the use of equivalent circuit networks to model the medium, [2]. In the case of dispersive dielectric medium modelled by a first-order Debye approximation, the frequency domain permittivity function is:

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega\tau_0} \qquad (10)$$

This corresponds to a RC circuit shown in Fig. 1. The state-equation can be obtained directly from the differential equation describing (9):

$$(\varepsilon_s - 1)\frac{dv_y}{dt} + \tau_0(\varepsilon_\infty - 1)\frac{d^2v_y}{dt^2} = i_y + \tau_0\frac{di_y}{dt} \qquad \text{(EQ 11)}$$

or, from the equivalent circuit, Fig. 1:

$$C1 = 2\Delta t(\varepsilon_\infty - 1)$$
$$C2 = 2\Delta t(\varepsilon_s - \varepsilon_\infty) \qquad (12)$$
$$R = \frac{\tau_0}{2\Delta t(\varepsilon_s - \varepsilon_\infty)}$$

resulting in the state-equation:

$$\frac{d}{dt}[v_u] = [A]\,[v_u] + [B]\,[v_{au}]$$
$$\left[v^i_{au}\right] = [C]\,[v_u] + [D]\,[v^r_{au}] \qquad \text{(EQ 13)}$$

where u=x,y,z and:

$$[A] = \begin{bmatrix} -\left(\frac{Y_r}{C1} + \frac{1}{RC1}\right) & \frac{1}{RC1} \\ \frac{1}{RC2} & -\frac{1}{RC2} \end{bmatrix} \qquad [B] = \begin{bmatrix} 2\frac{Y_r}{C1} \\ 0 \end{bmatrix} \text{(EQ 14)}$$

$$[C] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad [D] = \begin{bmatrix} -1 \end{bmatrix}$$

and:

$$[v_u] = \begin{bmatrix} v_u(t) \\ v_{un}(t) \end{bmatrix} \qquad [v^r_{au}] = [v^r_{au}(t)] \qquad (15)$$

The state-equations may be solved with one of the stable discretization schemes for the differential operator:

a) Backwards Euler:

$$[P] = ([U] - \Delta t[A])^{-1}$$
$$[T] = ([U] - \Delta t[A])^{-1}\Delta t[B] \qquad (16)$$
$$[x(t+\Delta t)] = [P]\,[x(t)] + [T]\,[u(t)]$$

where [U] is the identity matrix.

b) Approximate trapezoidal:

$$[P] = ([U] - \Delta t[A])^{-1}([U] + \Delta t[A])$$
$$[T] = ([U] - \Delta t[A])^{-1}\Delta t[B]$$
$$\left[u\left(t+\frac{\Delta t}{2}\right)\right] = \frac{[u(t+\Delta t)] + [u(t)]}{2} \qquad (17)$$
$$[x(t+\Delta t)] = [P]\,[x(t)] + [T]\left[u\left(t+\frac{\Delta t}{2}\right)\right]$$

NUMERICAL RESULTS

This technique was validated by comparing SCN TLM results for a dielectric-filled isotropic waveguide and calculating the cutoff frequency of an anisotropic waveguide. Both examples simulated a WR-28 (7.112 mm by 3.556 mm) waveguide using regular mesh with a discretization of 24 x 12 x 4.

In the first case, the waveguide was filled with a dielectric with ε_r of 2.22 and the results were obtained using the usual TLM and the state-variable formulation as shown in Fig. 2. It can also be seen that although the backwards Euler scheme is lossy there is no change in the central frequency.

In the second case a sapphire substrate was used as a dielectric. The permittivity tensor is:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_u\cos^2\varphi + \varepsilon_v\sin^2\varphi & \frac{1}{2}(\varepsilon_u - \varepsilon_v)\sin 2\varphi & 0 \\ \frac{1}{2}(\varepsilon_u - \varepsilon_v)\sin 2\varphi & \varepsilon_v\cos^2\varphi + \varepsilon_u\sin^2\varphi & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \qquad (18)$$

where the dielectric was sapphire (ε_u = 9.34 and ε_v = 11.49), [7].

The optical axis lies on the xy plane, rotated by an angle φ with respect to the x axis.The problem was calculated for φ of 0º, 45º and 90º using backwards

Euler discretization. The comparison between the exact and calculated results is shown in Table - I.

CONCLUSION

The technique presented in this paper can be used to model general constitutive relationships and requires few modifications to a TLM program. A general description of the medium relationships was obtained with equivalent node sources and the state-variable approach. The technique was validated by comparison with stub-loaded SCN results and exact solutions for the anisotropic case. Good agreement was observed in both cases.

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REFERENCES

1. W.J.R. Hoefer, "The transmission line matrix method - theory and applications", *IEEE Trans. Microwave Theory and Tech.*, vol. 33, Oct. 1985.
2. L. de Menezes, W.J.R. Hoefer, "Modelling frequency dependent dielectrics in TLM", *IEEE AP-S Symposium Digest*, pp. 1140-1143, Jun. 1994.
3. L. de Menezes, W.J.R. Hoefer, "Modelling nonlinear dispersive media in 2D-TLM", *24th European Microwave Conference Proceedings*, pp.1739-1744, Sept. 1994.
4. R. DeCarlo *Linear Systems - A State-Variable Approach with Numerical Implementation*, Prentice-Hall, 1989.
5. P. Naylor and R. Ait-Sadi, "Simple method for determining 3-D TLM nodal scattering in nonscalar problems," *Electronic Letters*, vol. 28, pp. 2353-2354, Dec. 3, 1992.
6. P.B. Johns, "A Symmetrical Condensed Node for the TLM Method", *IEEE Trans. Microwave Theory and Tech.*, vol. 35, n. 4, pp. 370-377, Apr. 1987.
7. N.G. Alexoupoulos, "Integrated-circuits structures on anisotropic substrates", *IEEE Trans. Microwave Theory and Tech.*, vol. 33, n. 10, pp. 847-881, Oct. 1985.

Table 1: - Comparison of Analytical and Calculated Results for the sapphire example.

Analytical Cutoff Frequency	Axis Angle	SCN - TLM	Error (%)
6.2221 GHz	0^o	6.21 GHz	0.19
6.5354 GHz	45^o	6.57 GHz	-0.53
6.9012 GHz	90^os	6.90 GHz	0.02

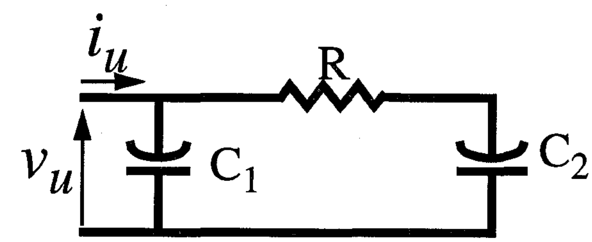

Fig. 1 Equivalent circuit network in the u (u=x,y,z) direction for the case of a dispersive dielectric medium modelled by a Debye approximation.

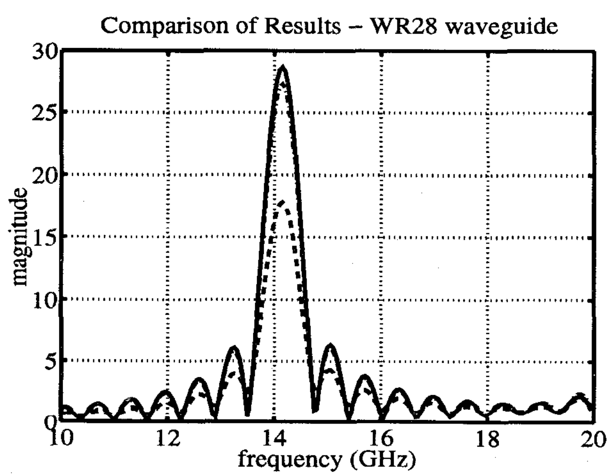

Fig. 2 Comparison of a WR 28 waveguide filled with dielectric ε_r = 2.22- cutoff frequencies. Solid line - stub-loaded SCN-TLM, dashed line - State-variable equations (backwards Euler discretization), dashed and dotted line - State-variable equations (approximate trapezoidal discretization).